A globally diagonalizable α^2-dynamo operator, SUSY QM and the Dirac equation

2007 J. Phys. A: Math. Theor. 40 F169

(http://iopscience.iop.org/1751-8121/40/5/F04)

IOP FTC
INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) F169–F176 doi:10.1088/1751-8113/40/5/F04

FAST TRACK COMMUNICATION

A globally diagonalizable α^2-dynamo operator, SUSY QM and the Dirac equation

Uwe Günther[1], **Boris F Samsonov**[2] **and Frank Stefani**[1]

[1] Research Center Dresden-Rossendorf, POB 510119, D-01314 Dresden, Germany
[2] Physics Department, Tomsk State University, 36 Lenin Avenue, 634050 Tomsk, Russia

E-mail: u.guenther@fzd.de, samsonov@phys.tsu.ru and f.stefani@fzd.de

Received 15 November 2006
Published 17 January 2007
Online at stacks.iop.org/JPhysA/40/F169

Abstract
A new class of semi-analytically solvable MHD α^2-dynamos is found based on a global diagonalization of the matrix part of the dynamo differential operator. Close parallels to SUSY QM are used to relate these models to the Dirac equation and to extract non-numerical information about the dynamo spectrum.

PACS numbers: 02.30.Tb, 03.65.Db, 03.65.Ge, 11.30.Pb, 47.65.Md, 91.25.Cw
Mathematics Subject Classification: 47B50, 46C20, 47A11, 81Q60, 34B07, 76W05

The magnetic fields of planets, stars and galaxies are maintained by dynamo effects in electrically conducting fluids or plasmas. A crucial role in the qualitative understanding of the basic dynamics of dynamos is played by various toy models which allow for a semi-analytical study. In this respect, the spherically symmetric α^2-dynamo [1, 2] plays a role similar to the harmonic oscillator in quantum mechanics (QM). But even this model can be analytically described only in a very few cases—like, e.g., for constant α-profiles (the α-profile $\alpha(r)$ acts as an effective potential in the dynamo operator) or in the case of idealized boundary conditions [3] (mimicking a perfect external conductor [4] or the limit of high angular mode numbers $l \gg 1$ [5]).

Here, we are going to present another class of highly simplifying α^2-dynamo models based on a global diagonalizability of the matrix part of the dynamo differential operator. For this purpose we relax the very rigid boundary conditions (BCs) at the surface of the dynamo maintaining fluid and replace it by a combination of a strongly localized α embedded in a conducting surrounding [6] and Dirichlet BCs at infinity. This allows us to use not only the Krein space symmetry properties of the operator [3, 7, 8] (closely related to $\mathcal{PT}$-symmetric QM [9]) but also to uncover deep relations to super-symmetric (SUSY) QM (for a recent review see [10]) and to map the dynamo eigenvalue problem into a set of two 2×2 stationary Dirac equations. Similar to [6], we will find a close relation between overcritical dynamo states and bound states in an associated QM model. For a certain parameter value the diagonalization technique breaks down and we develop a perturbative approach to describe the

system behaviour in the vicinity of the corresponding operator Jordan structure. The spectral reality properties of overcritical dynamo states are discussed. Finally, we interpret our findings in terms of a special link between the radial components of the electromotive force and the induced currents.

The subject of our analysis is the eigenvalue problem of a spherically symmetric α^2-dynamo [2] in its simplified and unitarily re-scaled form [7, 8]

$$(\partial_r K \partial_r + M - \lambda I)\,\Phi(r) = 0, \qquad \Phi(0) = 0, \qquad \Phi(\infty) = 0. \tag{1}$$

The matrix structure is encoded in

$$K = I - \alpha\sigma_-, \tag{2}$$

$$M = -K\frac{l(l+1)}{r^2} + \alpha\sigma_+, \tag{3}$$

where I is the 2×2 unit matrix and $\sigma_\pm$ denote the nilpotent matrices $\sigma_\pm = \frac{1}{2}(\sigma_x \pm \mathrm{i}\sigma_y)$ with $\sigma_{x,y,z}$ being the usual Pauli matrices. The system (1) describes the coupled poloidal and toroidal components of the l-modes of the magnetic field in a mean-field α^2-dynamo model with helical turbulence function (α-profile) $\alpha(r)$. We assume this α-profile real-valued, bounded, sufficiently smooth and exponentially decreasing for large $r \gg 1$. The latter assumption allows us to consider the α-profile approximatively as localized and at the same time to relax the otherwise rigidly imposed physical BCs at a given fluid/plasma surface[3] (set, e.g., at $r = 1$) replacing them by Dirichlet BCs at $r \to \infty$. Such an approach will make underlying structural links to quantum-mechanical setups transparent and will provide the Krein-space self-adjointness of the eigenvalue problem (1) similar to that of models with idealized (Dirichlet BCs) at fixed $r = 1$ (see [3, 7, 8]).

Our first goal is in diagonalizing the matrix structure of the 2×2 matrix differential operator in (1). For this purpose we use a two-step procedure, which consists of a replacement of the dependent variable by a Kummer–Liouville-type transformation

$$\Phi(r) = P^{-1}(r)\,\Xi(r) \tag{4}$$

to remove $\alpha\sigma_-$ from K in the derivative term and an afterwards performed coordinate-independent (global) similarity transformation to diagonalize the remaining matrix potential. The matrix P can be found to be a square root of K,

$$K = P^2, \qquad P = \begin{pmatrix} 1 & 0 \\ -\frac{\alpha}{2} & 1 \end{pmatrix}, \tag{5}$$

and yields the following equation for $\Xi(r)$:

$$\Xi''(r) - \frac{l(l+1)}{r^2}\,\Xi(r) + V(r)\,\Xi(r) = 0, \tag{6}$$

where

$$V(r) = \begin{pmatrix} \frac{1}{2}\alpha^2(r) - \lambda & \alpha(r) \\ \frac{1}{2}\alpha''(r) + \frac{1}{4}\alpha^3(r) - \alpha(r)\lambda & \frac{1}{2}\alpha^2(r) - \lambda \end{pmatrix}. \tag{7}$$

The equation system (6) can be globally decoupled provided the eigenvectors of the matrix (7) are r-independent. It is not difficult to see that this is possible if the function $\alpha(r)$ satisfies the equation

$$\alpha''(r) + \tfrac{1}{2}\alpha^3(r) - a^2\alpha(r) = 0 \tag{8}$$

[3] The physics of MHD dynamos is discussed, e.g., in [1, 2].

with a an arbitrary real constant. The general solution to this equation is expressed in terms of an elliptic integral which under the additional requirement $|\alpha(r \to \infty)| \to 0$ reduces to

$$\alpha(r) = \frac{2a}{\cosh[a(r - r_0)]} \tag{9}$$

with ar_0 as an integration constant. For α-profiles (9) the matrix part of the dynamo operator can be diagonalized—except for the special case $\lambda = \frac{1}{2}a^2$ when it is similar to a 2×2 Jordan block. Under the diagonalization $V \mapsto U^{-1}VU$ equation (6) splits into the following decoupled pair of differential equations (DEs) for the components (F_+, F_-) of the vector $U^{-1}\Xi = (F_+, F_-)^T$:

$$\left[-\partial_r^2 + \frac{l(l+1)}{r^2} - \frac{1}{2}\alpha^2 \mp \varepsilon\alpha\right] F_\pm = -\lambda F_\pm \tag{10}$$

with $\varepsilon = \left(\frac{1}{2}a^2 - \lambda\right)^{1/2}$ or $\varepsilon = -\left(\frac{1}{2}a^2 - \lambda\right)^{1/2}$ and the diagonalizing matrix U given by

$$U = \begin{pmatrix} 1 & 1 \\ \varepsilon & -\varepsilon \end{pmatrix}. \tag{11}$$

We note that in equations (8) and (10) the parameter a is inessential and can be eliminated by re-scaling $\varepsilon = a\tilde{\varepsilon}, r = x/a, \lambda = a^2\tilde{\lambda}, \alpha = a\tilde{\alpha}$. Further on, we will work in 'a units' what is equivalent to setting $a = 1$ and identifying $r = x, \varepsilon = \tilde{\varepsilon}, \lambda = \tilde{\lambda}, \alpha = \tilde{\alpha}$. Apart from (10) we will also use reshaped versions of these equations (obtained by substitution of $\lambda = \frac{1}{2} - \varepsilon^2$) which take the form of quadratic pencils in the auxiliary spectral parameter ε,

$$\left[-\partial_x^2 + \frac{l(l+1)}{x^2} - \frac{1}{2}\alpha^2 + \frac{1}{2} \mp \varepsilon\alpha - \varepsilon^2\right] F_\pm = 0, \qquad \alpha = \frac{2}{\cosh(x - x_0)}, \tag{12}$$

supplemented by the Dirichlet BCs $F_\pm(x = 0) = F_\pm(x = \infty) = 0$.

In the special case $\varepsilon = 0$, i.e. for $\lambda = \lambda_J := \frac{1}{2}$, the diagonalization matrix U (see (11)) becomes singular and the system (6) assumes the upper triangular (Jordan-type) form

$$\begin{pmatrix} \partial_x^2 - V_0 & -V_1 \\ 0 & \partial_x^2 - V_0 \end{pmatrix} \begin{pmatrix} \Xi_1 \\ \Xi_0 \end{pmatrix} = 0 \tag{13}$$

with Ξ_0, Ξ_1 as components of the vector $\Xi = (\Xi_1, \Xi_0)^T$ and the potential terms given by $V_0 = l(l+1)x^{-2} - \frac{1}{2}(\alpha^2 - 1), V_1 = -\alpha$.

We start our investigation of equations (10), (12) by noting that for $l \neq 0$ they are DEs of non-Fuchsian type and therefore their solutions cannot be expressed in terms of ordinary special functions. A certain simplification occurs for the monopole case $l = 0$. Then (10), (12) constitute a particular type of Heun's equations having three different finite regular singularities (see, e.g., [11]) and solutions expressible in terms of Heun's functions. A corresponding analysis will be presented elsewhere.

Here we are going to use the fact that equations (10), (12) are closely related to the exactly solvable stationary Schrödinger equation

$$H_1\phi = E\phi, \qquad H_1 = -\partial_x^2 - \tfrac{1}{2}\alpha^2 \tag{14}$$

with H_1 well known as superpartner[4] of the trivial Hamiltonian $H_0 = -\partial_x^2$

$$LH_0 = H_1L, \qquad L = -\partial_x + w', \qquad w = \partial_x \ln u, \qquad u = \cosh(x - x_0), \tag{15}$$

$$L^\dagger L = H_0 + 1, \qquad LL^\dagger = H_1 + 1, \qquad (H_0 - E_f)u = 0, \qquad E_f = -1 \tag{16}$$

[4] The potential $-\alpha^2/2 = -2/\cosh^2(x - x_0)$ is a x_0-shifted modified Pöschl–Teller [12] and reduced Rosen–Morse potential [13] and also known as 'one-soliton potential well' in KdV theory.

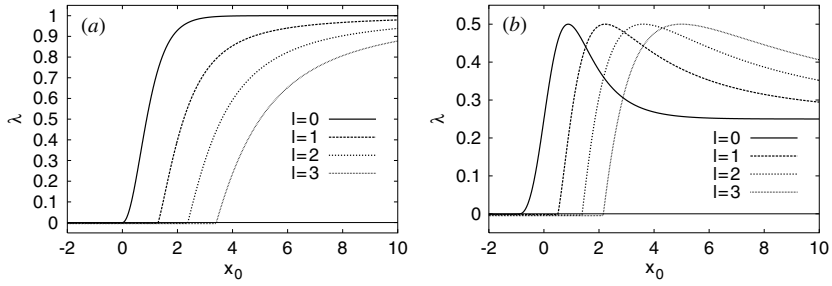

Figure 1. Spectra $\lambda(x_0)$ for the reduced system $[-\partial_x^2 + l(l+1)/x^2 - \alpha^2/2]F_\pm = -\lambda F_\pm$ (*a*) and for the complete problem (10) (*b*) in the case of angular mode numbers $l = 0, 1, 2, 3$. For numerical reasons the Dirichlet BC has been imposed at the large distance $x = 100$.

so that for $E > E_f = -1$ the solutions on the half-line $x \in [0, \infty)$ are simply given as

$$\phi_\pm = L\, \mathrm{e}^{\pm\kappa x} = [\mp\kappa + \tanh(x - x_0)]\, \mathrm{e}^{\pm\kappa x}, \qquad \kappa = \sqrt{-E}. \tag{17}$$

Imposing the Dirichlet BC $\phi(x = 0) = 0$ and exponential decay on these solutions makes H_1 an essentially self-adjoint operator with a single discrete level $E(x_0) = -\tanh^2(x_0) \in (-1, 0)$ for $x_0 > 0$ and with the half-line $E \geqslant 0$ as its continuous spectrum. The additional dynamo-related constraint $\phi(x \to \infty) \to 0$ selects then the bound state (BS) as a relevant solution. With this information at hand on the spectrum of the exactly solvable Schrödinger equation (14), we are now well prepared to present a qualitative discussion of the interrelated spectra of the dynamo eigenvalue problems (10).

A first piece of information can be extracted from (10) by neglecting for a moment the potential terms $l(l+1)/x^2$ and $\mp\varepsilon\alpha$. In this case (10) structurally coincides with (14) and we can identify $E = -\lambda$. This means that due to the physical constraint $\phi(x \to \infty) \to 0$ and its implication $E \in (-1, 0)$ the model necessarily describes overcritical dynamo regimes[5] $\lambda > 0$.

In order to extract further information, we proceed with the familiar QM model and extend H_1 by the centrifugal potential $l(l+1)/x^2$. This potential acts as a repulsive barrier in the vicinity of the origin $x = 0$ and for small x_0 it overcompensates the effect of the attractive one-soliton potential well $-\alpha^2/2 = -2/\cosh^2(x - x_0)$ with centre at $x = x_0$. As a result, no BS can exist for small x_0. When x_0 is increased beyond a certain l-dependent critical value the effect of the repulsive barrier will become sufficiently weak and the BS level will re-appear from the lower boundary $E = 0$ of the continuous spectrum and move down towards the lower boundary $E \to E_f = -1$ of the BS band. The situation is illustrated in figure 1(*a*) showing the sign-inverted picture for $\lambda = -E$.

It remains to clarify the role of the energy dependent part $\mp\varepsilon\alpha = \mp\left(\frac{1}{2} - \lambda\right)^{1/2}\alpha$ of the potential. Due to the specific square-root coupling this term might, in general, become complex-valued. Hence, the main question to answer is whether the spectrum will remain purely real (as the numerical results in figure 1(*b*) indicate) or whether and under which conditions it might become complex.

A partial clarification can be achieved by transforming the pencil equations (12) into equivalent Dirac equations. For this purpose we use a more general and slightly reshaped version of the SUSY factorization technique (16) (cf [14]), factorizing instead of H_1 the Hamiltonian

[5] A dynamo in its kinematic regime is called overcritical when it has a positive growth rate $\mathrm{Re}(\lambda) > 0$. This is in contrast to so-called undercritical regimes $\mathrm{Re}(\lambda) < 0$ with decaying (dissipating) magnetic field (see, e.g., [1, 2]).

$$H_{2,l} = -\partial_x^2 + \frac{l(l+1)}{x^2} - \frac{1}{2}\alpha^2 + \frac{1}{2} = L^\dagger L, \tag{18}$$

$$L = -\partial_x + w, \qquad L^\dagger = \partial_x + w, \qquad w = u'/u, \tag{19}$$

which has a continuous spectrum for $E \geqslant 1/2$ and a BS at some $E < 1/2$. The factorization (18) allows us to rewrite the pencils (12) as

$$[L^+ L \mp \varepsilon\alpha - \varepsilon^2]F_\pm = 0. \tag{20}$$

Denoting now $F_\pm =: \psi_1$ and assuming for $\varepsilon \neq 0$

$$L\psi_1 =: \varepsilon\psi_2 \tag{21}$$

equation (20) can be expressed as

$$L^+\psi_2 - (\pm\alpha + \varepsilon)\psi_1 = 0. \tag{22}$$

From the explicit form (19) of the operators L and L^+ we see that equations (21) and (22) are nothing but one-dimensional Dirac systems in their general representation (see, e.g., [15])

$$H\Psi = \varepsilon\Psi, \qquad H = \gamma\partial_x + V \tag{23}$$

with[6]

$$\Psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \qquad \gamma = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \qquad V = \begin{pmatrix}\mp\alpha & w\\ w & 0\end{pmatrix}. \tag{24}$$

For the factorization (18) to work, the function $u(x)$ should be a solution of the equation $H_{2,l}u(x) = 0$, i.e. an eigenfunction at the factorization energy $E_f = 0$. At the same time it should be nodeless on the positive semiaxis providing in this way a superpotential $w(x)$ which is regular. According to an implication of the 'oscillation' theorem (see, e.g., [16]) a real eigenfunction of the operator (18) can be nodeless only if its eigenvalue is located below the ground state level if the latter exists or below the lower bound of the continuous spectrum otherwise. Applied to our configuration with $E_f = 0$ this means that u is nodeless as long as the BS (ground state) energy $E(x_0) > E_f$. Comparison of (18) with (12) shows that $E_f = 0$ coincides with the energy of the Jordan configuration (13) at $\varepsilon = 0$ so that for $x_0 < x_J$ the solution u is nodeless and therefore w continuous. Together with the Dirichlet BC $\psi_1(x = 0) = 0$ and the easily verified boundedness of $|\psi_2(x = 0)| = |\varepsilon^{-1}L\psi_1|_{x=0} \leqslant C < \infty$ for $\varepsilon \neq 0$ this makes an adapted version of theorem III.7.1 from [15] applicable: if the coefficients of a Dirac system are continuous functions in any finite interval of the positive semi-axis then the Dirac operator H defined by the differential system (23) and the boundary condition $\psi_1(0)\cos\delta + \psi_2(0)\sin\delta = 0$ (with δ an arbitrary real number) is self-adjoint provided its domain of definition is $\psi_1, \psi_2 \in \mathcal{L}^2(0,\infty)$ with $L\psi_1 \in \mathcal{L}^2(0,\infty)$, $L^+\psi_2 - \alpha\psi_1 \in \mathcal{L}^2(0,\infty)$. This means that the Dirac system (23) supplemented with the Dirichlet BC at the origin, and with it the dynamo problem, has a purely real spectrum for $x_0 < x_J$.

In the case of $x_0 > x_J$ any real-valued solution to equation $H_{2,l}u = 0$ has a node on the positive semi-axis so that the superpotential $w(x)$ has a pole at some $x > 0$ and the theorem is no longer applicable. A circumvention of this problem might consist of a complex linear combination $u(x) = u^{(1)}(x) + \mathrm{i}cu^{(2)}(x)$, $c \in \mathbb{R}$ of two real-valued linearly independent solutions $u^{(1)}$ and $u^{(2)}$ of equation $H_{2,l}u = 0$. Such a function u has no nodes on the positive semi-axis, but the superpotential w becomes a complex-valued function. The spectrum of such a Dirac operator needs a special analysis which will be presented elsewhere.

[6] With the help of a gauge transformation $\Psi = A\Theta$ (see, e.g., [15]) which does not affect the zero boundary conditions, the Dirac equations (23) can be easily transformed into their canonical forms corresponding to a combination of scalar and pseudoscalar fields and describing the movement of massless particles.

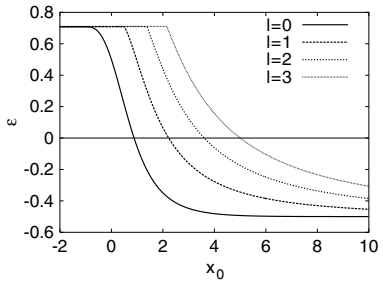

Figure 2. Spectrum in terms of $\varepsilon(x_0)$ for angular mode numbers $l = 0, 1, 2, 3$.

Here we proceed with a general qualitative analysis. We start from the observation that for small real $\lambda \leqslant 1/2$ the square root ε is real. Introducing an auxiliary parameter $b \in \mathbb{R}$ and replacing the potential term $\mp\varepsilon\alpha$ in (10) by $b\alpha$ the quadratic pencils (12) in ε reduce to an auxiliary linear eigenvalue problem in a b-dependent $\lambda(x_0, b)$ with additional constraints $b = \mp\varepsilon = \mp(1/2 - \lambda)^{1/2}$. The existence problem of the pencil solutions can be easily studied graphically in the (b, λ)-plane. A solution exists if the plot of the numerically obtained $\lambda(x_0, b)$ for given x_0 has an intersection point with the graphics of one of the constraints $b = b(\lambda)$. The corresponding analysis shows that for the branch $\varepsilon \geqslant 0$ one has a BS with $(F_+ \not\equiv 0, F_- \equiv 0)$ for $x_0 < x_J$ and with $(F_+ \equiv 0, F_- \not\equiv 0)$ for $x_0 > x_J$. Due to the invariance of (10) and (12) under the sign (branch) change $(\varepsilon, F_\pm) \mapsto (-\varepsilon, F_\mp)$ this can be re-interpreted as a single solution $(F_+ \not\equiv 0, F_- \equiv 0)$ with $\varepsilon(x_0 < x_J) > 0$ and $\varepsilon(x_0 > x_J) < 0$. The latter interpretation is confirmed by a direct numerical analysis (see figure 1(*b*)) of equations (10) (cross-checked by numerics on the original matrix-operator problem (1)) and by the graphics of $\varepsilon(x_0)$ in figure 2. Although a rigorous and complete analytical proof of the result for all $x_0 \in \mathbb{R}, l \geqslant 0$ is still missing, we now present a perturbative treatment of an $l = 0$ model in the vicinity of the non-diagonalizable operator configuration (13) at $\varepsilon = 0$ which explains the numerical results at least locally.

For $l = 0$ the Jordan chain equations (13) have the form

$$\left[-\partial_x^2 - \alpha^2/2 + 1/2\right] \Xi_0 = 0, \qquad \left[-\partial_x^2 - \alpha^2/2 + 1/2\right] \Xi_1 = \alpha \Xi_0 \tag{25}$$

and due to their relation to the eigenvalue problem (14), $(H_1 - E)\phi = 0$, $E = -1/2$ they are exactly solvable. From (17) we immediately read off that for BSs

$$\Xi_0 = C_0 \phi_-, \qquad \kappa = 2^{-1/2}, \qquad C_0 = \text{const.} \tag{26}$$

At the same time we find the value of the parameter $x_0 = x_J$ for which the Jordan structure occurs: $E(x_J) = -1/2 = -\tanh^2(x_J)$ yields $x_J = \operatorname{arctanh}(2^{-1/2})$. The second (associated) BS component Ξ_1 is easily obtained by standard techniques for inhomogeneous ODEs (see, e.g., [17])

$$\Xi_1 = C_1 \phi_- + \frac{C_0}{W}\left[\phi_+ \int_0^x \alpha \phi_-^2 \, \mathrm{d}x' - \phi_- \int_0^x \alpha \phi_+ \phi_- \, \mathrm{d}x'\right], \tag{27}$$

with W denoting the constant Wronskian $W \equiv W(\phi_+, \phi_-) = -2^{-1/2}$. In the limit $x \to \infty$ the product $\phi_+ \int_0^x \alpha \phi_-^2 \, \mathrm{d}x'$ diverges so that the BS condition $\Xi_1(\infty) = 0$ implies $C_0 = 0$ and the full Jordan chain solution reads simply $\Xi = (\Xi_1, \Xi_0)^T = (C_1 \phi_-, 0)^T$. Comparing the chain equations (25) at $\Xi_0 = 0$ with the pencil equations (12), i.e. with

$$\left[-\partial_x^2 - \tfrac{1}{2}\alpha^2 + \tfrac{1}{2} \mp \varepsilon\alpha - \varepsilon^2\right] F_\pm = 0, \tag{28}$$

shows that for BSs equations (25) can be interpreted as an effective smooth limit of (28) at $\varepsilon \to 0$. Hence, a BS perturbation theory can be constructed simply on the decoupled scalar equations (28) alone. As a small perturbation parameter we choose the x_0-distance $\delta = x_0 - x_J$ from the Jordan configuration and expand $\alpha = \alpha_J - \alpha'_J\delta + \cdots$, $\varepsilon = e_1\delta + e_2\delta^2 + \cdots$, $F_\pm = \Xi_1 + \chi_\pm\delta + \cdots$, where $\alpha_J = 2/\cosh(x - x_J)$ and $e_1, e_2, \ldots$, are coefficients to be defined from the perturbation scheme. This yields the defining equation for the first-order corrections $\chi_\pm$

$$\left[-\partial_x^2 - \alpha^2/2 + 1/2\right]\chi_\pm = -g_1\Xi_1, \qquad g_1 := \alpha_J[\alpha'_J \mp e_1] \tag{29}$$

with solutions

$$\chi_\pm = C_\pm\phi_- + \frac{C_1}{W}\left[\phi_- \int_0^x g_1\phi_+\phi_-\,\mathrm{d}x' - \phi_+\int_0^x g_1\phi_-^2\,\mathrm{d}x'\right]. \tag{30}$$

The BS condition $\chi_\pm(x \to \infty) \to 0$ can only be fulfilled if $\int_0^\infty g_1\phi_-^2\,\mathrm{d}x' = 0$. Hence, it fixes the parameter

$$e_1 = \pm\frac{\int_0^\infty \alpha_J\alpha'_J\phi_-^2\,\mathrm{d}x'}{\int_0^\infty \alpha_J\phi_-^2\,\mathrm{d}x'} = \mp\frac{1}{2} \tag{31}$$

and with it $\varepsilon = \mp\delta/2$. Knowing that for fixed x_0 the spectral parameter ε is the same in the equations for F_+ and F_- we conclude that $\varepsilon = \mp\delta/2$ can be valid only for one of the signs and that therefore it acts as a selection rule. Full compatibility with the numerical results and figure 2 is established by choosing the F_+-related BS for $\varepsilon > 0$ and $x_0 < x_J$, i.e. for $\delta < 0$, so that $\varepsilon = -\delta/2$ holds for all sufficiently small $\delta \in (-c_1, c_2)$ and provides a smooth connection between the $\varepsilon > 0$ and $\varepsilon < 0$ branches. At the same time it excludes a BS for the solution F_-.

Inspection of the recurrence algorithm for the higher order corrections shows that at each order the highest-order coefficient e_k enters its defining equation only linearly so that no square roots are involved which could produce complex-valued contributions. Together with the reality of all other ingredients $(\phi_\pm, \alpha_J, \alpha'_J, \ldots)$ of these recurrence equations we conclude that $\mathrm{Im}(e_k) = 0, \forall\, k \in \mathbb{Z}^+$ and no complex-valued BS-ε can emerge from an BS within the convergence region of the series $\varepsilon = \sum_{k=1}^\infty e_k\delta^k$. In this way we found an argumentation complementary to the Dirac equation based technique for $x_0 < x_J$. Another argument explaining the reality of the BS eigenvalue follows from the fact that the dynamo operator (1) with Dirichlet BCs is necessarily self-adjoint in a Krein space [3, 7, 8], and, hence, a spectral real-to-complex transition requires two spectral branches of different Krein-space types to coalesce at some point in parameter space. Once locally only a single BS exists there is also no chance for a BS-related spectral phase transition to complex eigenvalues within the convergence region of $\varepsilon = \sum_{k=1}^\infty e_k\delta^k$. The question of whether complex-conjugate BS eigenvalue pairs might split off from the continuum remains still open.

Finally we interpret the obtained solution behaviour $F_+ \not\equiv 0, F_- \equiv 0$ in terms of a special link between the magnetic field components of the dynamo. According to [1, 2] (cf also the appendix of [7]) the poloidal and toroidal components of the lth angular and nth radial modes of the multipole expanded fields $\mathbf{B}_p^{(l,n)}$ and $\mathbf{B}_t^{(l,n)}$ are given by $\mathbf{B}_p^{(l,n)} = -\nabla \times (\mathbf{r} \times \nabla)\, F_1^{(l,n)}(r,\theta)$, $\mathbf{B}_t^{(l,n)} = -\mathbf{r} \times \nabla F_2^{(l,n)}(r,\theta)$ with scalar functions $F_{1,2}^{(l,n)}(r,\theta) = r^{-1}\Phi_{1,2}^{(l,n)}(r)Y_l^0(\theta)$ built from spherical harmonics $Y_l^0(\theta) = \sqrt{(2l+1)/(4\pi)}\,P_l(\cos\theta)$ and the solution components $\Phi_{1,2}^{(l,n)}$ of problem (1). In our case of only one BS solution the n-dependence reduces to a single term $\Phi_{1,2}^{(l,1)}$. Expressing the components $F_\pm$ with the help of equations (4) and (11) in terms of $\Phi_{1,2}^{(l,1)}$

$$F_\pm = \pm\frac{r}{2\varepsilon}\left[\Phi_2^{(l,1)} - \left(\frac{\alpha}{2} \mp \varepsilon\right)\Phi_1^{(l,1)}\right] \tag{32}$$

and using the inverted relations [1] between the original magnetic field strength $\mathbf{B}^{(l,n)}$ and the corresponding scalar functions $F_{1,2}^{(l,n)}$, $\mathbf{r}\cdot\mathbf{B}^{(l,n)} = -l(l+1)F_1^{(l,n)}$, $\mathbf{r}\cdot(\nabla\times\mathbf{B}^{(l,n)}) = -l(l+1)F_2^{(l,n)}$, we arrive for $F_- \equiv 0$ after multiplying (32) with $-l(l+1)Y_l^0$ at

$$-l(l+1)Y_l^0 F_- = -\frac{r}{2\varepsilon}\mathbf{r}\cdot\left[\mu\mathbf{j}^{(l,1)} - \left(\frac{\alpha}{2}+\varepsilon\right)\mathbf{B}^{(l,1)}\right] \equiv 0. \tag{33}$$

(The relation $\mathbf{B} = \mu\mathbf{H}$ and one of the Maxwell equations, $\nabla\times\mathbf{H} = \mathbf{j}$, have been used.) Equation (33) has to be interpreted as special link between the induced current $\mathbf{j}$ and the spectrally shifted electromotive force $\alpha\mathbf{B}$ in the present dynamo model. However, it can be shown that the field is not a Beltrami field $\nabla\times\mathbf{B} \neq \beta\mathbf{B}$.

Acknowledgments

We thank K-H Rädler for useful comments on Beltrami fields. BFS is partially supported by grants RFBR-06-02-16719, SS-5103.2006.2. He also thanks the Research Center Dresden-Rossendorf for hospitality during his stays in Dresden and the financial support under contract MHD 3-06. UG has been supported by the German Research Foundation DFG, grant GE 682/12-3.

References

[1] Moffatt H K 1978 *Magnetic Field Generation in Electrically Conducting Fluids* (Cambridge: Cambridge University Press)
[2] Krause F and Rädler K-H 1980 *Mean-Field Magnetohydrodynamics and Dynamo Theory* (Berlin: Akademie-Verlag, Oxford, Pergamon)
[3] Günther U and Kirillov O 2006 *J. Phys. A: Math. Gen.* **39** 10057 (*Preprint* math-ph/0602013)
[4] Proctor M R E 1977 *Astron. Nachr.* **298** 19
[5] Znojil M and Günther U 2006 Dynamics of charged fluids and $1/\ell$ perturbation expansions *Preprint* math-ph/0610055
[6] Meinel R 1989 *Astron. Nachr.* **310** 1
[7] Günther U and Stefani F 2003 *J. Math. Phys.* **44** 3097 (*Preprint* math-ph/0208012)
[8] Günther U, Stefani F and Znojil M 2005 *J. Math. Phys.* **46** 063504 (*Preprint* math-ph/0501069)
[9] Bender C M and Boettcher S 1998 *Phys. Rev. Lett.* **80** 5243 (*Preprint* physics/9712001)
[10] Mielnik B and Rosas-Ortiz O 2004 *J. Phys. A: Math. Gen.* **43** 10007
[11] Ronveaux A (ed) 1995 *Heun's Differential Equations* (Oxford: Oxford University Press)
[12] Grosche C and Steiner F 1998 *Handbook of Feynman Path Integrals* (Berlin: Springer)
[13] De Lange O L and Raab R E 1991 *Operator Methods in Quantum Mechanics* (Oxford: Clarendon)
[14] Antonowicz A and Fordy A P 1989 *Commun. Math. Phys.* **124** 465
[15] Kostyuchenko A G and Sargsyan I S 1979 *Distribution of Eigenvalues. Selfadjoint Ordinary Differential Operators* (Moscow: Nauka)
[16] Berezin F A and Shubin M A 1991 *The Schrödinger Equation* (Dordrecht: Kluwer)
[17] Polyanin A D and Zaitsev V F 1995 *Handbook of Exact Solutions for Ordinary Differential Equations* (Boca Raton, FL: CRC Press)